Energy Storage Hand Exerciser for Stroke Rehabilitation

Paul Lazarescu
University of California, Irvine

August 10, 2011

Abstract

This article describes the detailed steps in formulating the dynamic equation of the flywheel-based energy storage hand exerciser.

1 Assumptions

1. The system can be modeled as a four-bar linkage.
2. The friction is located only at the pivot point of the hand.
3. The mass of bar l_2 is negligible when compared to the mass of the flywheel and the mass of the user's hand.
4. The torsional spring used to model the user's hand is always in tension, as a stroke victim continuously exerts force inwards.

2 Kinematics

Describe the motion in terms of a position vector, eliminating α and solving for $\theta(\phi)$.

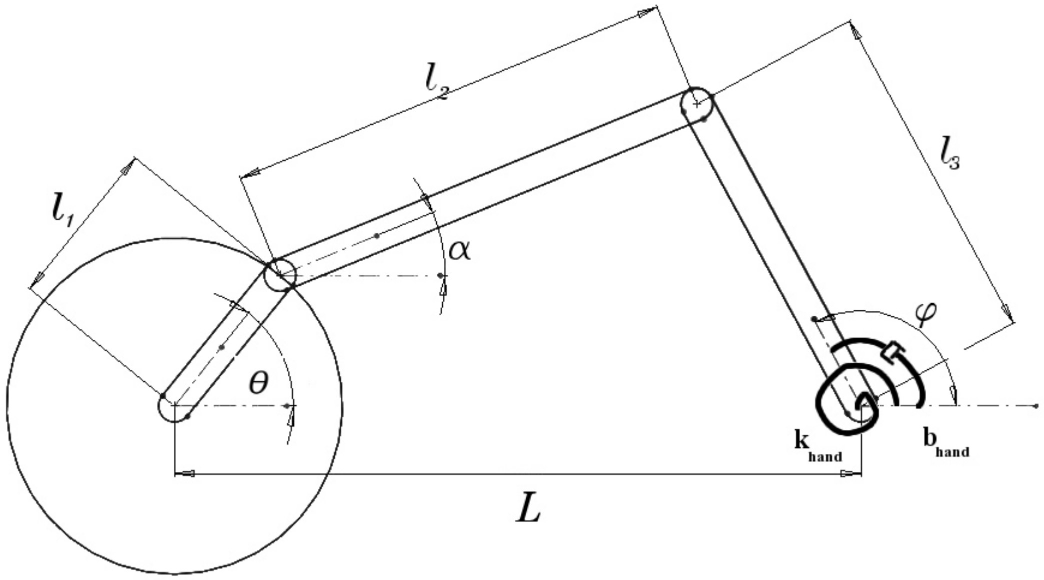

Figure 1: A model of the hand exerciser system

Using the loop equations to come up with a system of equations describing the system:

$$l_1 \sin\theta + l_2 \sin\alpha - l_3 \sin\phi = 0 \tag{1}$$

$$-l_1 \cos\theta - l_2 \cos\alpha + l_3 \cos\phi + L = 0 \tag{2}$$

If we put the α terms on the left side, to eliminate:

$$l_2 \sin\alpha = -l_1 \sin\theta + l_3 \sin\phi \tag{3}$$

$$l_2 \cos\alpha = -l_1 \cos\theta + l_3 \cos\phi + L \tag{4}$$

Squaring Eq.(1) and Eq. (2) and combining terms, we get:

$$l_2^2 = L^2 + l_1^2 + l_3^2 - 2Ll_1 \cos\theta + 2Ll_3 \cos\phi - 2l_1 l_3 \cos\phi \cos\theta - 2l_1 l_3 \sin\phi \sin\theta \tag{5}$$

This can be arranged in the following form:

$$k_1(\phi) \sin\theta + k_2(\phi) \cos(\theta) + k_3(\phi) = 0 \tag{6}$$

Where:

$$k_1(\phi) = -2l_1 l_3 \sin\phi \tag{7}$$

$$k_2(\phi) = 2l_1(L - l_3 \cos\phi) \tag{8}$$

$$k_3(\phi) = L^2 + l_1^2 - l_2^2 + l_3^2 + 2Ll_3 \cos\phi \tag{9}$$

Once in the above form, we can make use of *Freudenstein's Equation* to solve for θ as a function of ϕ. We define:

$$t = \tan\frac{\theta}{2} \tag{10}$$

$$\sin\theta = \frac{2t}{1+t^2} \tag{11}$$

$$\cos\theta = \frac{1-t^2}{1+t^2} \tag{12}$$

Substitute the above into Eq. (6) gives a quadratic equation in terms of t:

$$(k_3 - k_2)t^2 + (2k_1)t + (k_3 + k_2) = 0 \tag{13}$$

Solving for t:

$$t = \frac{-k_1 \pm \sqrt{k_1^2 + k_2^2 - k_3^2}}{k_3 - k_2} \tag{14}$$

Now we can get θ as a function of ϕ by substituting Eq. (10) into Eq. (14):

$$\theta(\phi) = 2\arctan\frac{-k_1 \pm \sqrt{k_1^2 + k_2^2 - k_3^2}}{k_3 - k_2} \tag{15}$$

3 Lagrangian formulation

We set up the Lagrangian, $\mathcal{L} = T - V$. The total kinetic energy of this system is the sum of the kinetic energies of each member:

$$T = KE_1 + KE_2 + KE_3 \tag{16}$$

However, according to Assumption 3., $KE_2 = 0$. Therefore the total Kinetic Energy of this system is:

$$T = \frac{1}{2}I_1\dot{\theta}^2 + \frac{1}{2}I_3\dot{\phi}^2 \tag{17}$$

The potential energy of this system is the energy stored in the torsional spring:

$$V = \frac{1}{2}k\phi^2 \tag{18}$$

The Lagrangian is therefore:

$$\frac{1}{2}I_1\dot{\theta}^2 + \frac{1}{2}I_3\dot{\phi}^2 - \frac{1}{2}k\phi^2 \tag{19}$$

4 Equation of Motion

Lagrange's equation is:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i \tag{20}$$

Where q_i is the generalized coordinate. In this case, we choose ϕ, the angle of the user's hand, to be the generalized coordinate. First, we solve for the generalized force:

$$Q_i = -k\phi - b\dot{\phi} + \tau_{ext} \tag{21}$$

Plugging in our value of the Lagrangian into Lagrange's equation, and simplifying, we get our equation of motion:

$$\frac{1}{2} I_1 \frac{d}{dt}\left(\frac{d}{d\dot{\phi}}\dot{\theta}^2\right) + I_3\ddot{\phi} - \frac{1}{2} I_1 \frac{d}{d\phi}\dot{\theta}^2 + 2k\phi + b\dot{\phi} = \tau_{ext} \tag{22}$$

To simplify this for the final equation of motion, we would utilize Eq. (15) and Eq. (7)-(9).

References

[1] Chin Pei Tang, "Lagrangian dynamic formulation of a four-bar mechanism with minimal coordinates," University of Texas at Dallas, 2006.